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Frequency Equation of a Curved Beam using the Phase-closure Principle
위상폐합원리를 이용한 곡선보의 진동수 방정식

Nansukusa Mirembe Sarah* and Hyun Woo Park†
미렘베 세라 난수쿠사* · 박 현 우†

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ABSTRACT

This paper presents a simplified frequency equation that predicts the modal frequencies of a curved beam. In particular, the simplified frequency equation is derived for a frequency range within which one pair of propagating wave motions and two pairs of evanescent wave motions exist on the curved beam. All incident evanescent wave motions are assumed to be negligible at both ends of the beam. The phase-closure principle is applied to a curved beam with varying support conditions. First, the wave reflection coefficients for the curved beam are calculated, after which the phases of the reflection coefficients are applied using the phase-closure principle to derive the frequency equation. Then, the Newton-Raphson method is employed to compute the modal frequencies from the frequency equation. The proposed frequency equation is validated with numerical results for varying support conditions and span angles.

요 약

이 논문은 곡선보의 모드 진동수를 예측하기 위해 단순화된 진동수 방정식을 제시한다. 특히, 한쌍의 전달 파동과 두쌍의 소멸 파동이 존재하는 곡선보의 진동수 범위에서 단순화된 진동수 방정식을 유도한다. 보의 양쪽 지점으로 입사되는 소멸 파동은 무시할 수 있다고 가정한다. 위상폐합원리를 다양한 지점 조건을 가지는 곡선보에 적용한다. 곡선보에서 파 반사 계수를 먼저 계산한 후 파 반사에 의한 위상변화를 위상폐합원리에 적용하여 진동수 방정식을 유도한다. 진동수 방정식으로부터 모드 진동수를 계산하기 위해 뉴턴-랩슨법을 적용한다. 제안된 주파수 방정식은 다양한 지지 조건 및 스팬 각도에 대한 수치 해석 결과로 검증한다.

1. Introduction

Vibration of curved beams has been the subject of numerous studies(1~4). Curved beams are commonly used in engineering structures such as aircraft structures, bridges, and modern electric machine parts(5~6).

† Corresponding Author ; Member, Department of ICT Integrated Ocean Smart Cities, Dong-A University, Professor
E-mail : hwpark@donga.ac.kr
* Former graduate student, Department of Civil Engineering, Dong-A University, Student

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Various methods are applied to study the dynamics of curved beam structures. The Rayleigh-Ritz method was used to formulate a frequency equation to derive the lowest natural frequencies of circular arcs with varying boundary conditions(7). Chidamparam and Leissa utilized the Galerkin method to obtain exact natural frequencies for extensional and inextensional loaded circular arches(8). The transfer matrix method was employed to investigate in-plane and out-of-plane frequencies of plane curved beams while accounting for shear deformation, rotatory inertia and extension of neutral axis(9). Issa et al.(10) applied the dynamics stiffness method to the effects of shear deformation and rotatory inertia on extensional free vibrations to determine natural frequencies of continuous curved beams. The widely used method, the finite element method (FEM), was employed to study the static and free vibration of linear beam elements for curved beams (11). Most of the mentioned methods become extremely burdensome when large number of spans are used(12). The FEM is also computationally expensive when a refined mesh is required for predicting the high-frequency modal frequency which is sensitive to an incipient crack on the beam(13).

The wave approach is another method employed to analyze curved beams. It is a concise and systematic approach used to analyze structures since it easily allows efficient variation of the geometry and size of complex structures. Studies have applied the wave approach to predict the modal frequencies of curved beams(14~16). Natural frequencies of curved beams can be obtained while using a common wave technique of formulating propagation, reflection and transmission characteristics of waves.

This wave technique is based on a principle known as the phase-closure principle(17). The phase-closure principle is also called the wave-train closure principle(18). The phase-closure principle is one where natural frequencies occur when the total phase change of a complete circuit of a wave propagating around a system is an integer multiple of 2π.

Mead(17) applied the phase-closure principle to formulate an exact frequency equation for a single span fixed beam while using propagating and evanescent waves. Tang et al.(19) used a simplified frequency equation to calculate the natural frequencies of a uniform rod and beam with nonlinear stiffness boundaries. The phase of the reflection coefficients at each boundary were applied with the phase-closure principle to obtain the natural frequencies of the system.

This study aims to formulate a simplified frequency equation for a curved beam using the phase-closure principle to predict high-frequency modal frequencies. In particular, the simplified frequency equation is derived for a frequency range within which one pair of propagating wave motions and two pairs of evanescent wave motions exist on the curved beam.

The reflection coefficients at varying support conditions are calculated. The phases are obtained from the reflection coefficients. Then, the phases of each support condition are applied with the phase-closure principle to determine the modal frequencies of a single span curved beam. The modal frequencies for the beam calculated from the proposed frequency equation are then presented and compared with those from the matrix determinant method. The advantages of the proposed method are discussed as well.

2. Spectral Solutions of a Curved Beam

The governing equations of motion are considered from Fig. 1 where N, V and M are axial force, shear force and bending moment, respectively(20). Note that effects of rotary inertia, shear deformations and damping are neglected.

The normalized equations of motion in Eq. (1) are modified from non-dimensional variables and parameters given as

$$k^2\frac{\partial^3}{\partial\theta^3}\left(u-\frac{\partial w}{\partial\theta}\right)-\left(w+\frac{\partial u}{\partial\theta}\right)=k^2\frac{\partial^2 w}{\partial t^2} \tag{1a}$$

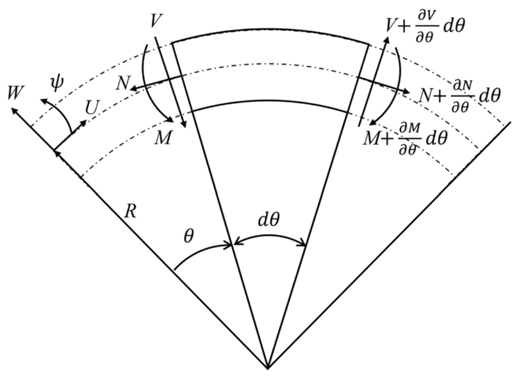
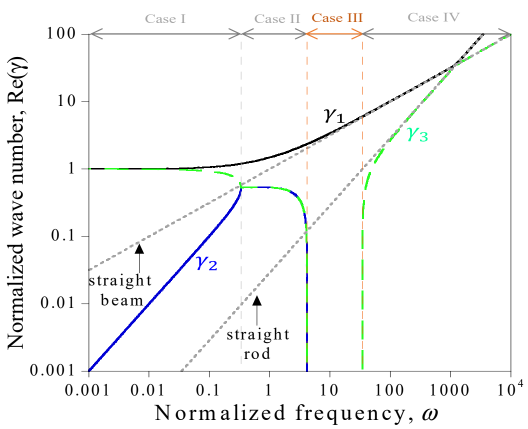


Fig. 1 Differential element of a curved beam and sign conventions

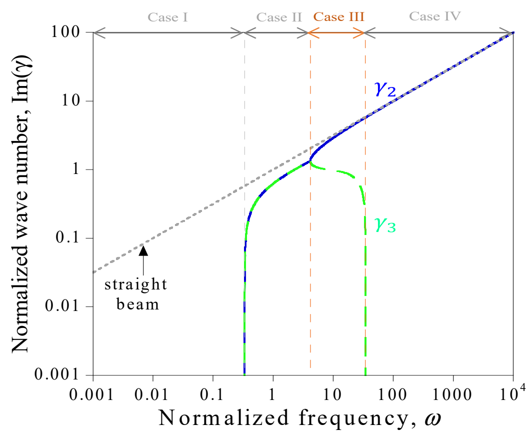
$$k^2\frac{\partial^2}{\partial\theta^2}\left(u-\frac{\partial w}{\partial\theta}\right)+\frac{\partial}{\partial\theta}\left(w+\frac{\partial u}{\partial\theta}\right)=k^2\frac{\partial^2 u}{\partial t^2} \tag{1b}$$

$$u=\frac{U}{R},\quad w=\frac{W}{R},\quad t=\frac{T}{T_0},\quad T_0=R^2\sqrt{\frac{\rho A}{EI}},\quad k^2=\frac{I}{AR^2} \tag{1c}$$

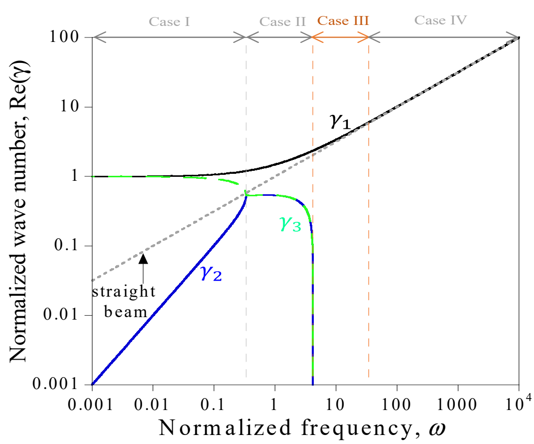
where u, w and t are the non-dimensional tangential, radial displacements and time variable respectively. R is the constant radius of curvature for the given range of angle θ, T_0 is the characteristic time, ρ is the mass density, A is the cross-sectional area, E is Young's modulus, I is second



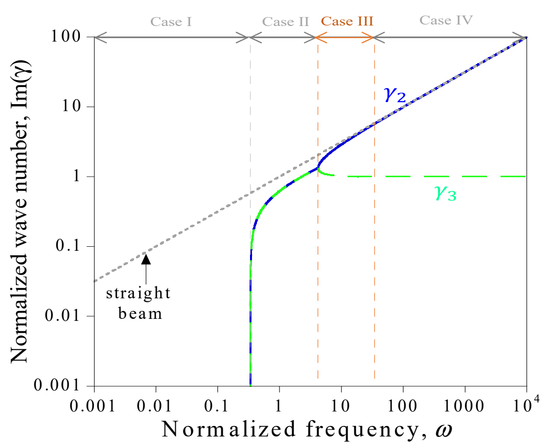
(a) Real branches (k = 0.0289)



(b) Imaginary branches (k = 0.0289)



(c) Real branches (k = 0)



(d) Imaginary branches (k = 0)

Fig. 2 Dispersion curves of a curved beam for extentioned case (k = 0.0289) and for inextensional case (k = 0), the dispersion curves of straight beam and rod are presented as well for comparison purposes moment of inertia. Furthermore, curvature parameter k is defined as the ratio of the radius of gyration of the cross-section ($\sqrt{I/A}$) to the radius of curvature R.

Time harmonic solutions were assumed to solve the governing equations which are provided as

$$w(\theta,t)=C_w e^{i(\gamma\theta-\omega t)} \tag{2a}$$

$$u(\theta,t)=C_u e^{i(\gamma\theta-\omega t)} \tag{2b}$$

where γ and ω are non-dimensional wavenumber normalized by $1/R$ and non-dimensional frequency normalized by $1/T_0$, respectively.

The dispersion equation of the wave number γ obtained from the determinant of matrix of harmonic solutions is provided as

$$\gamma^6-(2+k^2\omega^2)\gamma^4+\{1-(1+k^2)\omega^2\}\gamma^2+(k^2\omega^2-1)\omega^2=0 \tag{3}$$

Figure 2 illustrates the dispersion curves obtained by solving Eq. (3) for both extentional and inextensional curved beams[20]. Four distinct wave motions which are depicted as from Case I to IV can be observed in Fig. 2. Among these four cases of wave motions, this study focuses on Case III in which one pair of propagating wave motions and two pairs of evanescent wave motions exist for both extensional and inextensional curved beams. The spectral solution for Case III can be expressed as follows:

$$w(\theta,t)=\begin{pmatrix} C_{w1}^{+}e^{-i\gamma_1\theta}+C_{w2}^{+}e^{-i\gamma_2\theta}+C_{w3}^{+}e^{-i\gamma_3\theta} \\ +C_{w1}^{-}e^{i\gamma_1\theta}+C_{w2}^{-}e^{i\gamma_2\theta}+C_{w3}^{-}e^{i\gamma_3\theta}\end{pmatrix}e^{-i\omega t} \tag{4}$$

where γ_1 is a positive real root while γ_2 and γ_3 are two negative roots obtained from Eq. (3).

The phase-closure principle is easily applied with Case III of wave motion since one pair of propagating wave motions is used to predict the natural frequencies of structure. The range of frequency for Case III wave motion is $4.164\langle\omega\langle\omega_c=34.641$.

3. Simplified Frequency Equation from the Phase-closure Principle

3.1 Reflection coefficient

The spectral solutions of normalized radial and tangential displacement are expressed as follows

$$w(\theta,t)=\hat{w}(\theta)e^{-i\omega t} \tag{5a}$$

$$u(\theta,t)=\hat{u}(\theta)e^{-i\omega t} \tag{5b}$$

where

$$\hat{w}(\theta)=C_{w1}^{+}e^{-i\gamma_1\theta}+C_{w2}^{+}e^{-i\gamma_2\theta}+C_{w3}^{+}e^{-i\gamma_3\theta} +C_{w1}^{-}e^{i\gamma_1\theta}+C_{w2}^{-}e^{i\gamma_2\theta}+C_{w3}^{-}e^{i\gamma_3\theta} \tag{5c}$$

$$\hat{u}(\theta)= C_{w1}^{+}\alpha_1e^{-i\gamma_1\theta}+C_{w2}^{+}\alpha_2e^{-i\gamma_2\theta}+C_{w3}^{+}\alpha_3e^{-i\gamma_3\theta} -C_{w1}^{-}\alpha_1e^{i\gamma_1\theta}-C_{w2}^{-}\alpha_2e^{i\gamma_2\theta}-C_{w3}^{-}\alpha_3e^{i\gamma_3\theta} \tag{5d}$$

The displacement and moment boundary conditions at a hinged support ($\theta=0$)are expressed as

$$\hat{w}\big|_{\theta=0}=0 \tag{6a}$$

$$\hat{u}\big|_{\theta=0}=0 \tag{6b}$$

$$\left.\frac{\partial\hat{u}}{\partial\theta}-\frac{\partial^2\hat{w}}{\partial\theta^2}\right|_{\theta=0}=0 \tag{6c}$$

Substituting Eqs. (5) into Eqs. (6) results in:

$$\hat{w}\big|_{\theta=0}=C_{w1}^{+}+C_{w2}^{+}+C_{w3}^{+}+C_{w1}^{-}+C_{w2}^{-}+C_{w3}^{-}=0 \tag{7a}$$

$$\hat{u}\big|_{\theta=0}=C_{w1}^{+}\alpha_1+C_{w2}^{+}\alpha_2+C_{w3}^{+}\alpha_3-C_{w1}^{-}\alpha_1-C_{w2}^{-}\alpha_2-C_{w3}^{-}\alpha_3=0 \tag{7b}$$

$$\left.\frac{\partial\hat{u}}{\partial\theta}-\frac{\partial^2\hat{w}}{\partial\theta^2}\right|_{\theta=0}=\beta_1C_{w1}^{+}+\beta_2C_{w2}^{+}+\beta_3C_{w3}^{+}+\beta_1C_{w1}^{-}+\beta_2C_{w2}^{-}+\beta_3C_{w3}^{-}=0 \tag{7c}$$

where $\alpha_i = C_u / C_w = (i\gamma_i(1+\gamma_i^2 k^2)/(\gamma_i^2(1+k^2)-k^2\omega^2)$ and $\beta_i = \gamma_i^2 - i\gamma_i\alpha_i$. Eqs. (7a) ~ (7c) are simplified as follows by assuming that all incident evanescent wave motions associated with C_{w2}^+, C_{w3}^+, $C_{w2}^+\alpha_2$, $C_{w3}^+\alpha_3$are egligible in $\hat{w}(\theta)$ and $\hat{u}(\theta)$.

$$\hat{w}\big|_{\theta=0} = C_{w1}^+ + C_{w1}^- + C_{w2}^- + C_{w3}^- = 0 \tag{8a}$$

$$\hat{u}\big|_{\theta=0} = C_{w1}^+\alpha_1 - C_{w1}^-\alpha_1 - C_{w2}^-\alpha_2 - C_{w3}^-\alpha_3 = 0 \tag{8b}$$

$$\left.\frac{\partial \hat{u}}{\partial \theta} - \frac{\partial^2 \hat{w}}{\partial \theta^2}\right|_{\theta=0} = \beta_1 C_{w1}^+ + \beta_1 C_{w1}^- + \beta_2 C_{w2}^- + \beta_3 C_{w3}^- = 0 \tag{8c}$$

Adding α_2 ×Eq. (8a) to Eq. (8b) results in

$$(\alpha_1+\alpha_2)C_{w1}^+ - (\alpha_1-\alpha_2)C_{w1}^- + (\alpha_2-\alpha_3)C_{w3}^- = 0 \tag{9}$$

Adding $-\beta_2$ ×Eq. (8a) to Eq. (8c) produces

$$(\beta_1-\beta_2)C_{w1}^+ + (\beta_1-\beta_2)C_{w1}^- - (\beta_2-\beta_3)C_{w3}^- = 0 \tag{10}$$

Adding $(\beta_2-\beta_3)$×Eq. (9) to $(\alpha_2-\alpha_3)$×Eq. (10) yields

$$(\beta_2-\beta_3)(\alpha_1+\alpha_2)C_{w1}^+ - (\beta_2-\beta_3)(\alpha_1-\alpha_2)C_{w1}^- + (\alpha_2-\alpha_3)(\beta_1-\beta_2)C_{w1}^+ + (\alpha_2-\alpha_3)(\beta_1-\beta_2)C_{w1}^- = 0 \tag{11}$$

Using Eq. (11), reflection coefficient r from incident propagating wave motion C_{w1}^+ to reflected propagating wave motion C_{w1}^- can be expressed as follows

$$C_{w1}^- = \frac{\{(\beta_2-\beta_3)(\alpha_1+\alpha_2)+(\alpha_2-\alpha_3)(\beta_1-\beta_2)\}}{\{(\beta_2-\beta_3)(\alpha_1-\alpha_2)-(\alpha_2-\alpha_3)(\beta_1-\beta_2)\}}C_{w1}^+ = r_{\text{hinged}}C_{w1}^+ \tag{12}$$

$(\beta_i-\beta_j)$ in Eq. (12) can be written as

$$\beta_i-\beta_j = -i\gamma_i q_i + i\gamma_j q_j = -i(\gamma_i q_i - \gamma_j q_j) \tag{13}$$

where $q_i = \alpha_i + i\gamma_i$.

Substituting Eq. (13) into Eq. (12) results in

$$r_{\text{hinged}} = \frac{(\alpha_1+\alpha_2)(\gamma_2 q_2-\gamma_3 q_3)+(\alpha_2-\alpha_3)(\gamma_1 q_1-\gamma_2 q_2)}{(\alpha_1-\alpha_2)(\gamma_2 q_2-\gamma_3 q_3)-(\alpha_2-\alpha_3)(\gamma_1 q_1-\gamma_2 q_2)} \tag{14}$$

The same procedure of obtaining the reflection coefficient is applied for fixed and free support conditions as respectively (elaborated in appendix A),

$$r_{\text{fixed}} = \frac{(\alpha_1+\alpha_2)(q_2-q_3)+(\alpha_2-\alpha_3)(q_1+q_2)}{(\alpha_1-\alpha_2)(q_2-q_3)-(\alpha_2-\alpha_3)(q_1-q_2)} \tag{15}$$

$$r_{\text{free}} = -\frac{(p_1\gamma_2 q_2+p_2\gamma_1 q_1)(p_2 q_3-p_3 q_2) - (p_2\gamma_3 q_3-p_3\gamma_2 q_2)(p_1 q_2-p_2 q_1)}{(p_1\gamma_2 q_2-p_2\gamma_1 q_1)(p_2 q_3-p_3 q_2) - (p_2\gamma_3 q_3-p_3\gamma_2 q_2)(p_1 q_2-p_2 q_1)} \tag{16}$$

where $p_i = \alpha_i + (i/\gamma_i)$.

3.2 Frequency equation

The phase-closure formula is applied to the curved beam shown in Fig. 3. The equations are derived as follows.

$$\int d\Gamma - \phi_L - \phi_R = 2c\pi \tag{17}$$

where $d\Gamma$, ϕ_L and ϕ_R are the phase shift of the propagating wave along the infinitesimal segment of the curved beam, the phase shifts due to wave re-

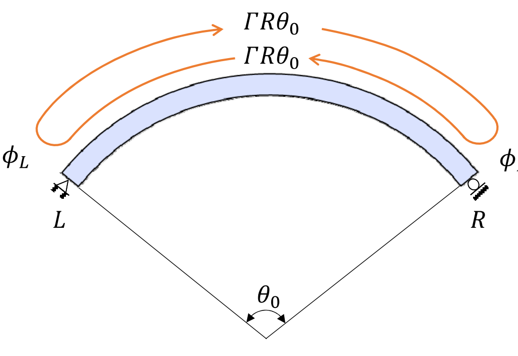


Fig. 3 Curved beam to illustrate the principle of phase closure

flections at the right and left end of the beam respectively while c is an arbitrary integer. Assuming a constant curvature along the whole length of the curved beam, Eq. (17) can be written as:

$$2\Gamma R\theta_0 - \phi_L - \phi_R = 2c\pi \tag{18}$$

where θ_0 is the span angle of the curved beam, respectively. Substituting $\Gamma = \gamma / R$ into Eq. (18) results in:

$$2\gamma\theta_0 - (\phi_L + \phi_R) = 2c\pi \tag{19}$$

Note that the ϕ_L and ϕ_R are determined by the phase of a reflection coefficient which is expressed in Eqs. (14) ~ (16) depending on the support types of the curved beam.

3.3. Numerical solutions of the frequency equation using Newton-Raphson (N-R) method

The numerical solutions of Eq. (19) are obtained using the Newton-Raphson method because the frequency equation is nonlinear with respect to non-dimensional angular frequency ω. The frequency equation Eq. (19) is rewritten as:

$$y = 2\gamma\theta_0 - (\phi_L + \phi_R) - 2c\pi = 0 \tag{19}$$

The modal angular frequency ω satisfying Eq. (20) is solved iteratively by linearizing Eq. (20) with respect to ω:

$$y_{j+1} \approx y_j + \frac{dy_j}{d\omega}\Delta\omega = 0 \tag{21}$$

where j denotes the number of the N-R iteration while y_j and $dy_j/d\omega$ represent $y|_{\omega=\omega_j}$ and $dy/d\omega|_{\omega=\omega_j}$, respectively. The solution increment $\Delta\omega$ is expressed as follows by using Eq. (21).

$$\Delta\omega = -\frac{y_j}{\frac{dy_j}{d\omega}} \tag{22}$$

The modal frequency is updated by adding the solution increment from Eq. (22):

$$\omega_{j+1} = \omega_j + \Delta\omega \tag{23}$$

The N-R iteration is repeated by using Eqs. (21) ~ (23) until the following termination criterion is satisfied:

$$\left|\frac{\omega_{j+1} - \omega_j}{\omega_{j+1}}\right| < \epsilon \tag{24}$$

where ϵ is termination tolerance (e.g. $\epsilon = 10^{-7}$)

The first order derivative y with respect to ω is expressed as:

$$\frac{dy}{d\omega} = \frac{dy}{d\gamma_1}\frac{d\gamma_1}{d\omega} \tag{25}$$

where γ_1 is a positive real root of the dispersion equation of a curved beam for case III region expressed as:

$$\gamma_1^6 - (2 + k^2\omega^2)\gamma_1^4 + \{1 - (1 + k^2)\omega^2\}\gamma_1^2 + (k^2\omega^2 - 1)\omega^2 = 0 \tag{26}$$

$d\gamma_1/d\omega$ in Eq. (25) is obtained by differentiating Eq. (26) with respect to ω:

$$\frac{d\gamma_1}{d\omega} = \frac{2k^2\omega\gamma_1^4 + 2(1 + k^2)\omega\gamma_1^2 - (4k^2\omega^3 - 2\omega)}{6\gamma_1^5 - 4(2 + k^2\omega^2)\gamma_1^3 + 2\{1 - (1 + k^2)\omega^2\}\gamma_1} \tag{27}$$

$dy/d\gamma_1$ in Eq. (25) is expressed as follows by differentiating Eq. (20) with respect to γ_1:

$$y' = 2\theta_0 - (\phi_L' + \phi_R') \tag{28}$$

where $'$ denotes the differential operator with respect to r_1.

Assuming that both support conditions at the left and right ends of the beam are identical, ϕ_L and ϕ_R in Eq. (20) are expressed as follows using the reflection coefficient r:

$$\phi_L = \phi_R = \arg(r) \tag{29}$$

where r is calculated by using Eqs. (14) ~ (16) depending on the types of the support conditions.

Decomposing r into a real part $\mathrm{Re}(r)$ and an imaginary part $\mathrm{Im}(r)$, Eq. (29) is rewritten as:

$$\tan\phi_L = \tan\phi_R = \frac{\mathrm{Im}(r)}{\mathrm{Re}(r)} \tag{30}$$

Differentiating Eq. (30) with respect to γ_1 results in

$$\phi_L' = \frac{\mathrm{Im}(r)'\mathrm{Re}(r) - \mathrm{Im}(r)\mathrm{Re}(r)'}{|r|^2} \tag{31}$$

Note that $\mathrm{Im}(r)'$ and $\mathrm{Re}(r)'$in Eq. (31) can be obtained through differentiating the reflection coefficient r with respect to γ_1.

$$r' = \mathrm{Re}(r)' + i\,\mathrm{Im}(r)' \tag{32}$$

where $\mathrm{Re}(r)' = \mathrm{Re}(r')$ and $\mathrm{Im}(r)' = \mathrm{Im}(r')$.

The first order derivative r' is obtained through differentiating Eqs. (14) ~ (16) depending on the type of the support condition. The detailed procedure to derive r' is provided in Appendix B.

4. Results and Discussions

The numerical solutions of the proposed frequency described in Section 3 are validated for both extensional and inextentional curved beams with varying support conditions and span angles. Table 1 provides the non-dimensional natural frequencies of a curved beam with span angles 90° and 180° for three support conditions. The results in Table 1

Table 1 Non-dimensional natural frequencies of a curved beam in case III region ($4.164 < \omega < 34.641$)

Span angle (degree)	BC*	Mode no.**	Extensional ($k^2 = 1/1200$)			Inextensional ($k = 0$)		
			Matrix determinant	Proposed method	Difference (%)	Matrix determinant	Proposed method	Difference (%)
90	H-H	1	13.71	13.11	4.380	13.76	13.27	3.566
		2	27.49	30.97	12.65	32.40	33.21	2.494
	F-F	1	-	5.798	-	-	5.876	-
		2	22.44	20.38	9.213	22.63	21.40	5.427
		3	28.11	33.01	17.41	-	-	-
	FR-FR	1	8.382	8.268	1.359	8.391	8.278	1.354
		2	23.89	23.90	0.03048	23.93	23.93	0.02940
180	H-H	1	6.887	6.963	1.105	6.923	6.992	0.9907
		2	13.90	13.82	0.6278	13.98	13.92	0.4307
		3	22.37	22.58	0.9461	22.82	22.89	0.2911
		4	33.40	32.24	3.474	33.93	33.87	0.1737
		5	33.70	34.51	2.400	-	-	-
	F-F	1	9.498	9.766	2.818	9.652	9.874	2.306
		2	17.70	17.41	1.675	17.92	17.76	0.9128
		3	25.64	26.51	3.367	27.52	27.69	0.6178
		4	34.58	33.20	4.001	-	-	-
	FR-FR	1	5.303	5.305	0.04609	5.309	5.311	0.04436
		2	11.10	11.10	0.00058	11.11	11.12	0.00093
		3	18.99	18.99	0.00021	19.02	19.02	0.00033
		4	28.91	28.91	0.00001	28.96	28.96	0.00006

* BC = boundary condition, H-H = hinged-hinged, F-F = fixed-fixed and FR-FR = free-free
** Mode number is based on the natural frequencies obtained from the proposed method in the ascending order

were compared with those obtained by the matrix determinant method which is described in appendix C.

As shown in Table 1, the frequency equation of this study reasonably predicted the natural frequencies of the curved beam. The relative error between predictions of the current study and the determinant matrix method is considerably small. For the extensional case, the error tends to fluctuate with increase in the mode number as compared to the inextensional case. The error in the inextensional case decreases as the mode number increases.

The respective comparison results for span angle 90° are provided in Fig. 4. The circle and the square represent the natural frequencies from the matrix determinant method and the proposed method in Table 1, respectively. The red and blue lines depict real and imaginary values of $C(\omega)$ in Eq. (C7) with respect to while the black line depict the equation residual of y in Eq. (20) with respect to ω.

It should be noted that substituting the roots of the proposed frequency equation in Eq. (20) into $C(\omega)$results in $\mathrm{Re}(C(\omega))=0$ and $\mathrm{Im}(C(\omega))\neq 0$ for all cases. This implies that the proposed frequency equation in which incident evanescent wave motions at both supports are neglected produces the roots satisfying $\mathrm{Re}(C(\omega))=0$ only. Therefore, the discrepancy of the natural frequencies between the proposed frequency equation and the matrix determinant method depends on the distance between the roots of $\mathrm{Re}(C(\omega))=0$ with $\mathrm{Im}(C(\omega))\neq 0$ and those

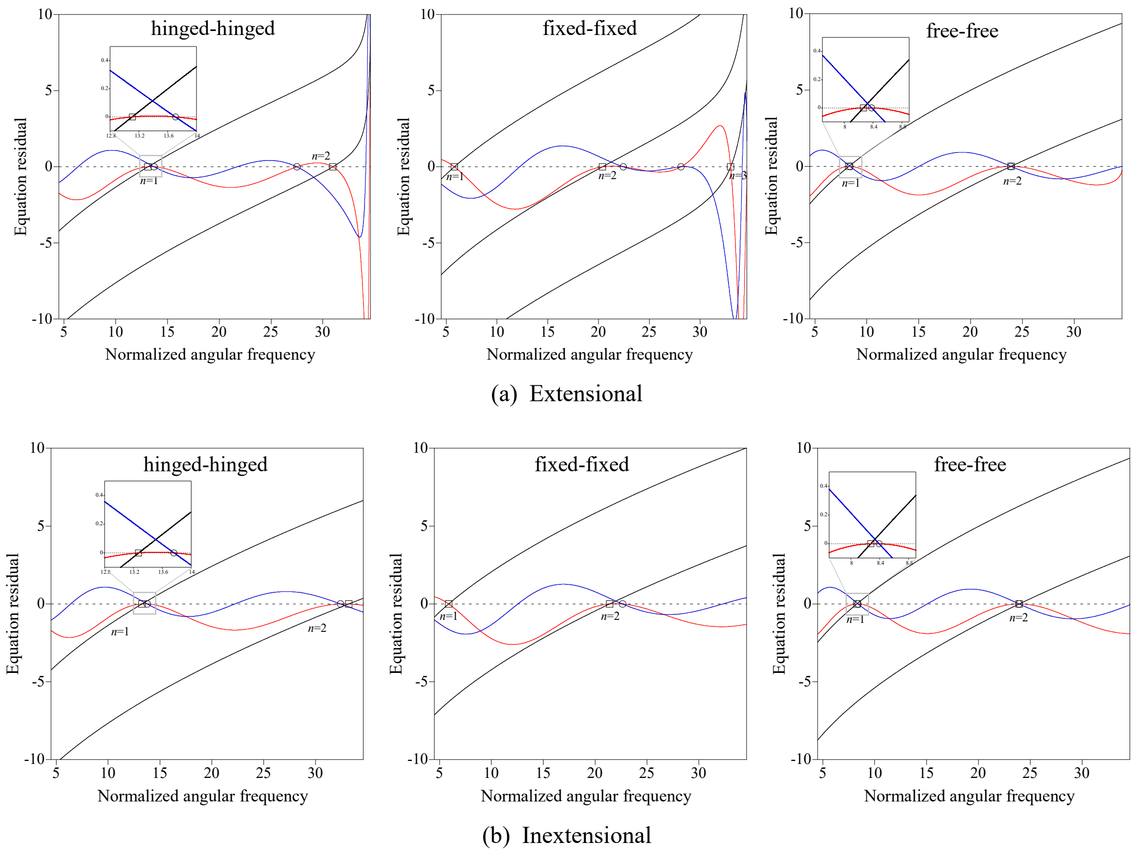


Fig. 4 Comparison of the natural frequencies between matrix determinant method and the proposed method for a curved beam with span angle 90° (circle: roots of $C(\omega)$in Eq. (C7), square: roots of in Eq. (20), red line: $\mathrm{Re}(C(\omega))$, blue line: $\mathrm{Im}(C(\omega))$, black line: y in Eq. (20))

of $\mathrm{Re}(C(\omega)) = 0$ with $\mathrm{Im}(C(\omega)) = 0$. For the extensional curved beam, the discrepancy becomes significant as approaches either the low cut-off frequency (i.e. 4.164) dividing case II and III regions or the high cut-off frequency (i.e. 34.641) dividing case III and IV regions in Figs. 2(a) ~ (b) due the appearance of incident evanescent wave motions. For the inextensional curved beam, the discrepancy becomes significant as approaches the low cut-off frequency dividing case II and III regions because there is no case IV region as shown in Figs. 2(c) ~ (d). The fictitious natural frequency ($n = 1$) appears for both extensional and inextensional fixed-fixed curved beams due to the incident evanescent wave motions near the low-cut off frequency. The relatively large errors in the second and the third natural frequencies of the extensional hinged-hinged and fixed-fixed curved beams can be attributed to the incident evanescent wave motions near the high cut-off frequency. Note that the errors in the natural frequencies of the inextensional curved beams decrease as increases regardless of the support conditions. This indicates that the incident evanescent wave motions vanish as the frequency increases in case III region.

Figure 5 presents the respective comparison results for span angle 180°. The meanings of all symbols and lines are identical to those used in Fig. 4. As the span angle is doubled, the errors in the first natural frequency significantly decreases compared

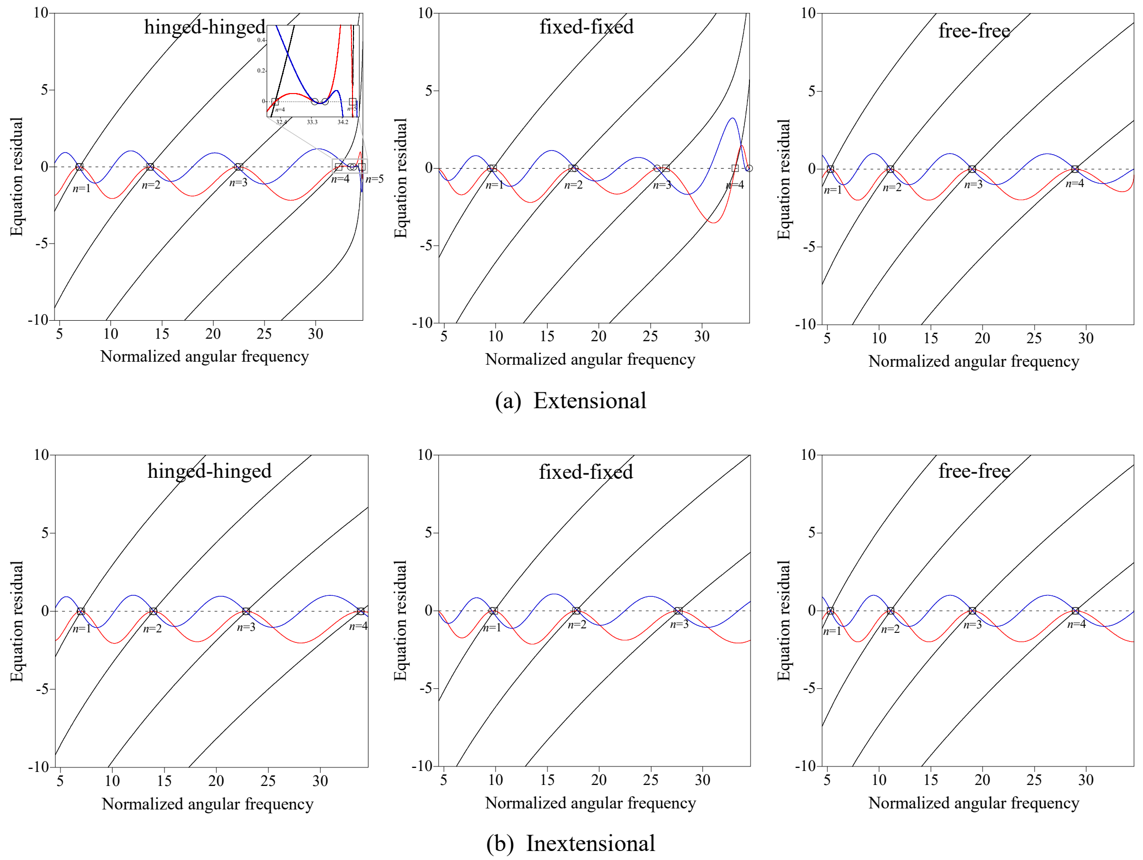


Fig. 5 Comparison of the natural frequencies between matrix determinant method and the proposed method for a curved beam with span angle 180° (circle: roots of $C(\omega)$ in Eq. (C7), square: roots of in Eq. (20), red line: $\mathrm{Re}(C(\omega))$, blue line: $\mathrm{Im}(C(\omega))$, black line: y in Eq. (20))

to those in Fig. 4 for both extensional and inextensional curved beams regardless of the support conditions. For higher natural frequencies near the high cut-off frequency, the error decreases significantly for the extensional curved beam compared to those in Fig. 4. In case of inextensional curved beams, the proposed frequency equation produces practically identical results from the matrix determinant method as increases.

The advantages of the proposed method over the matrix determinant method can be described in two aspects. First, y in the frequency equation [Eq. (20)] is a monotonically increasing function with respect to per mode number n which results in a single root per each as shown in Fig. 4 and Fig. 5. Second, y in Eq. (20) is a concave function in the vicinity of the root in most cases except that it exists near the high cut-off frequency. Therefore, the N-R method produces the converged solution within a few iterations as long as the initial guess is set near the low cut-off frequency (e.g. 5).

$C(\omega)$ in the frequency equation [Eq. (C7)] is a transcendental function with respect to in which multiple roots exist. In this case, the convergence of the N-R method highly depends on the initial guess which should be decided by trial and error. Therefore, searching for all roots in $C(\omega)$ may become a very laborious task.

This study has some limitations. Only propagating wave components were used and the incident evanescent wave motions were neglected. Therefore, the proposed frequency equation only provides approximate results for the natural frequencies. It also should be noted that the proposed method is complementary to the matrix determinant method because the proposed frequency equation can provide the reliable initial guess when the N-R method is adopted in the matrix determinant method.

5. Conclusions

This paper has proposed a simple closed-form frequency equation to predict the modal frequencies of a curved beam considering the phase-closure principle. The frequency equation is derived for a frequency range within which one pair of propagating wave motions and two pairs of evanescent wave motions exist. The reflection coefficients at each boundary were derived from which phases of reflection were obtained and applied to the phase-closure principle to determine the high-frequency modal frequencies of a curved beam.

The proposed frequency equation was validated for both extensional and inextentional curved beams with span angles 90° and 180° for three support conditions. The numerical solutions calculated from the proposed equation were comparable with those from the matrix determinant method. The interesting point is that every solution from the proposed frequency equation always makes only the real part of the frequency equation from the matrix determinant method zero while the corresponding imaginary part nonzero. Overall, the discrepancy of the natural frequencies between the proposed frequency equation and the matrix determinant method is attributed to the incident evanescent wave motions near the low and high cut-off frequencies of the case III region.

The relative error between the numerical solutions of the proposed frequency equation and the matrix determinant method decreased as the mode number increased especially for inextensional curved beams. Furthermore, the proposed frequency equation produces the converged solution within a few iterations as long as the initial guess is set near the low cut-off frequency. The formulated frequency equation is simple, easy and more straight forward in obtaining approximate natural frequencies of a curved beam.

The proposed method is complementary to the matrix determinant method because the proposed frequency equation can provide the reliable initial guess when the N-R method is adopted in the matrix determinant method.

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Appendix A

a) Fixed support condition

The displacement and rotational boundary conditions at fixed support ($\theta=0$) are expressed as

$$\hat{w}\big|_{\theta=0}=0 \tag{A1a}$$

$$\hat{u}\big|_{\theta=0}=0 \tag{A1b}$$

$$\left.\frac{\partial\hat{w}}{\partial\theta}-\hat{u}\right|_{\theta=0}=0 \tag{A1c}$$

Substituting Eqs. (5) into Eq. (A1) results in:

$$\begin{aligned}\hat{w}\big|_{\theta=0}=C_{w1}^{+}+C_{w2}^{+}+C_{w3}^{+}+C_{w1}^{-}\\+C_{w2}^{-}+C_{w3}^{-}=0\end{aligned} \tag{A2a}$$

$$\begin{aligned}\hat{u}\big|_{\theta=0}=C_{w1}^{+}\alpha_1+C_{w2}^{+}\alpha_2+C_{w3}^{+}\alpha_3-C_{w1}^{-}\alpha_1\\-C_{w2}^{-}\alpha_2-C_{w3}^{-}\alpha_3=0\end{aligned} \tag{A2b}$$

$$\begin{aligned}\left.\frac{\partial\hat{w}}{\partial\theta}-\hat{u}\right|_{\theta=0}=-\beta_1C_{w1}^{+}-\beta_2C_{w2}^{+}-\beta_3C_{w3}^{+}\\+\beta_1C_{w1}^{-}+\beta_2C_{w2}^{-}+\beta_3C_{w3}^{-}=0\end{aligned} \tag{A2c}$$

Equations. (A2a) ~ (A2c) are simplified as follows due to the assumption that all incident evanescent wave motions are negligible:

$$\hat{w}\big|_{\theta=0}=C_{w1}^{+}+C_{w1}^{-}+C_{w2}^{-}\ \ +C_{w3}^{-}=0 \tag{A3a}$$

$$\begin{aligned}\hat{u}\big|_{\theta=0}=C_{w1}^{+}\alpha_1-C_{w1}^{-}\alpha_1\\-C_{w2}^{-}\alpha_2-C_{w3}^{-}\alpha_3=0\end{aligned} \tag{A3b}$$

$$\begin{aligned}\left.\frac{\partial\hat{w}}{\partial\theta}-\hat{u}\right|_{\theta=0}=-\beta_1C_{w1}^{+}+\beta_1C_{w1}^{-}\\+\beta_2C_{w2}^{-}+\beta_3C_{w3}^{-}=0\end{aligned} \tag{A3c}$$

Adding $\alpha_2\times$Eq. (A3a) to Eq. (A3b) results in

$$(\alpha_1+\alpha_2)C_{w1}^{+}-(\alpha_1-\alpha_2)C_{w1}^{-}+(\alpha_2-\alpha_3)C_{w3}^{-}=0 \tag{A4}$$

Adding $-\beta_2$ × Eq. (A3a) to Eq. (A3c) produces

$$\begin{aligned}-(\beta_1+\beta_2)C_{w1}^{+}+(\beta_1-\beta_2)C_{w1}^{-}\\-(\beta_2-\beta_3)C_{w3}^{-}=0\end{aligned} \tag{A5}$$

Adding $(\beta_2-\beta_3)$ × Eq. (A4) to $(\alpha_2-\alpha_3)$ ×Eq. (A5) produces the reflection coefficient γ_{fixed} in Eq. (15) through some mathematical manipulation.

b) Free support condition

The displacement and rotational boundary conditions at free support ($\theta=0$)are expressed as

$$\left.\frac{\partial\hat{u}}{\partial\theta}+\hat{w}\right|_{\theta=0}=0 \tag{A6a}$$

$$\left.\frac{\partial^2\hat{u}}{\partial\theta^2}-\frac{\partial^3\hat{w}}{\partial\theta^3}\right|_{\theta=0}=0 \tag{A6b}$$

$$\left.\frac{\partial\hat{u}}{\partial\theta}-\frac{\partial^2\hat{w}}{\partial\theta^2}\right|_{\theta=0}=0 \tag{A6c}$$

Substituting Eqs. (5) into Eqs. (A6a) ~ (A6c) results in:

$$\begin{aligned}&\left.\frac{\partial\hat{u}}{\partial\theta}+\hat{w}\right|_{\theta=0}\\&=i\gamma_1p_1C_{w1}^{+}+i\gamma_2p_2C_{w2}^{+}+i\gamma_3p_3C_{w3}^{+}\\&\quad+i\gamma_1p_1C_{w1}^{-}+i\gamma_2p_2C_{w2}^{-}\ \ +i\gamma_3p_3C_{w3}^{-}=0\end{aligned} \tag{A7a}$$

$$\begin{aligned}&\left.\frac{\partial^2\hat{u}}{\partial\theta^2}-\frac{\partial^3\hat{w}}{\partial\theta^3}\right|_{\theta=0}\\&=-\gamma_1^2q_1C_{w1}^{+}-\gamma_2^2q_2C_{w2}^{+}-\gamma_3^2q_3C_{w3}^{+}+\gamma_1^2q_1C_{w1}^{-}\\&\quad+\gamma_2^2q_2C_{w2}^{-}+\gamma_3^2q_3C_{w3}^{-}=0\end{aligned} \tag{A7b}$$

$$\left.\frac{\partial\hat{u}}{\partial\theta}-\frac{\partial^2\hat{w}}{\partial\theta^2}\right|_{\theta=0} = \beta_1 C_{w1}^{+}+\beta_2 C_{w2}^{+}+\beta_3 C_{w3}^{+} + \beta_1 C_{w1}^{-}+\beta_2 C_{w2}^{-}+\beta_3 C_{w3}^{-}=0 \tag{A7c}$$

Equations (A7a) ~ (A7c) are simplified as follows due to the assumption that all incident evanescent wave motions are negligible:

$$\left.\frac{\partial\hat{u}}{\partial\theta}+\hat{w}\right|_{\theta=0} = i\gamma_1 p_1 C_{w1}^{+}+i\gamma_1 p_1 C_{w1}^{-}+i\gamma_2 p_2 C_{w2}^{-}+i\gamma_3 p_3 C_{w3}^{-} =0 \tag{A8a}$$

$$\left.\frac{\partial^2\hat{u}}{\partial\theta^2}-\frac{\partial^3\hat{w}}{\partial\theta^3}\right|_{\theta=0} =-\gamma_1^2 q_1 C_{w1}^{+}+\gamma_1^2 q_1 C_{w1}^{-}+\gamma_2^2 q_2 C_{w2}^{-}+\gamma_3^2 q_3 C_{w3}^{-} =0 \tag{A8b}$$

$$\left.\frac{\partial\hat{u}}{\partial\theta}-\frac{\partial^2\hat{w}}{\partial\theta^2}\right|_{\theta=0} = \beta_1 C_{w1}^{+}+\beta_1 C_{w1}^{-}+\beta_2 C_{w2}^{-}+\beta_3 C_{w3}^{-} =0 \tag{A8c}$$

Adding $\gamma_2^2 q_2$ ×Eq. (A8a) to $-i\gamma_2 p_2$ ×Eq. (A8b) results in

$$\left(i\gamma_1\gamma_2^2 p_1 q_2+i\gamma_2\gamma_1^2 p_2 q_1\right)C_{w1}^{+} +\left(i\gamma_1\gamma_2^2 p_1 q_2-i\gamma_2\gamma_1^2 p_2 q_1\right)C_{w1}^{-} -\left(i\gamma_2\gamma_3^2 p_2 q_3-i\gamma_3\gamma_2^2 p_3 q_2\right)C_{w3}^{-}=0 \tag{A9}$$

Adding β_2 ×Eq. (A8a) to $-i\gamma_2 p_2$ ×Eq. (A8c) produces

$$\left(i\gamma_1 p_1\beta_2-i\gamma_2 p_2\beta_1\right)C_{w1}^{+} +\left(i\gamma_1 p_1\beta_2-i\gamma_2 p_2\beta_1\right)C_{w1}^{-} -\left(i\gamma_2 p_2\beta_3-i\gamma_3 p_3\beta_2\right)C_{w3}^{-}=0 \tag{A10}$$

Adding $\left(i\gamma_2 p_2\beta_3-i\gamma_3 p_3\beta_2\right)$×Eq. (A9) to $\left(i\gamma_2\gamma_3^2 p_2 q_3-i\gamma_3\gamma_2^2 p_3 q_2\right)$×Eq. (A10) produces the reflection coefficient r_{free} in Eq. (16) through some mathematical manipulation.

Appendix B

The derivative of reflection coefficient with respect to γ_1 (r') can be expressed through the derivatives of rational functions as follows:

$$r'=\frac{U'V-UV'}{V^2} \tag{B1}$$

where U and V represent the numerator and the denominator of r which vary depending on the type of support condition as described in Eqs. (14) ~ (16). The derivatives of variables with respect to γ_1 needed to calculate Eq. (B1) are listed below:

a) Derivatives of γ_2 and γ_3 with respect to γ_1

$$\gamma'_2=\frac{1}{\gamma}\left\{k^2\omega\omega'-\gamma_1-\gamma_3\left(\frac{1}{\gamma_3^2-\gamma_2^2}z_1\right)\right\} \tag{B2a}$$

$$\gamma'_3=\frac{1}{\gamma_3^2-\gamma_2^2}z_1 \tag{B2b}$$

where

$z_1=\gamma_3\left(k^2\omega\omega'-\gamma_1\right)-\gamma_2\left((2\omega\omega'-4k^2\omega^3\omega'/2\gamma_1^2\gamma_2\gamma_3)-(\gamma_2\gamma_3/\gamma_1)\right)$ and ω' can be obtained from Eq. (27).

b) Derivatives of α_1, α_2 and α_3 with respect to γ_1

$$\alpha'_1=\frac{i\left(1+3\gamma_1^2k^2\right)\left\{\gamma_1^2(1+k^2)-k^2\omega^2\right\}}{\left\{\gamma_1^2(1+k^2)-k^2\omega^2\right\}^2} -\frac{\left\{2\gamma_1(1+k^2)-2k^2\omega\omega'\right\}i\gamma_1\left(1+\gamma_1^2k^2\right)}{\left\{\gamma_1^2(1+k^2)-k^2\omega^2\right\}^2} \tag{B3a}$$

$$\alpha'_2=\frac{i\left(\gamma'_2+3\gamma_2^2\gamma'_2k^2\right)\left\{\gamma_2^2(1+k^2)-k^2\omega^2\right\}}{\left\{\gamma_2^2(1+k^2)-k^2\omega^2\right\}^2} -\frac{\left\{2\gamma_2\gamma'_2(1+k^2)-2k^2\omega\omega'\right\}i\gamma_2\left(1+\gamma_2^2k^2\right)}{\left\{\gamma_2^2(1+k^2)-k^2\omega^2\right\}^2} \tag{B3b}$$

$$\alpha'_3=\frac{i\left(\gamma'_3+3\gamma_3^2\gamma'_3k^2\right)\left\{\gamma_3^2(1+k^2)-k^2\omega^2\right\}}{\left\{\gamma_3^2(1+k^2)-k^2\omega^2\right\}^2} -\frac{\left\{2\gamma_3\gamma'_3(1+k^2)-2k^2\omega\omega'\right\}i\gamma_3\left(1+\gamma_3^2k^2\right)}{\left\{\gamma_3^2(1+k^2)-k^2\omega^2\right\}^2} \tag{B3c}$$

c) Derivatives of p_1, p_2 and p_3 with respect to γ_1

$$p_1' = \alpha_1' - \frac{i}{\gamma_1^2} \tag{B4a}$$

$$p_2' = \alpha_2' - \frac{i\gamma_2'}{\gamma_2^2} \tag{B4b}$$

$$p_3' = \alpha_3' - \frac{i\gamma_3'}{\gamma_3^2} \tag{B4c}$$

d) Derivatives of q_1, q_2 and q_3 with respect to γ_1

$$q_1' = \alpha_1' + i \tag{B5a}$$

$$q_2' = \alpha_2' + i\gamma_2' \tag{B5b}$$

$$q_2' = \alpha_2' + i\gamma_2' \tag{B5c}$$

Appendix C

Figure C1 shows a general curved beam structure with boundaries L and R. The incident and reflected wave vector at boundaries are denoted as $\mathbf{w}_R^+$, $\mathbf{w}_R^-$, $\mathbf{w}_L^+$ and $\mathbf{w}_L^-$.

The relationships of the waves between the boundaries are given as

$$\mathbf{w}_L^+ = \mathbf{r}_L \mathbf{w}_L^- \tag{C1a}$$

$$\mathbf{w}_R^- = \mathbf{r}_R \mathbf{w}_R^+ \tag{C1b}$$

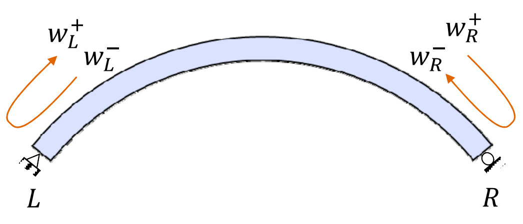

Fig. C1 Reflection of waves through a curved beam with constant curvature

$$\mathbf{w}_L^- = \mathbf{t}\mathbf{w}_R^- \tag{C1c}$$

$$\mathbf{w}_R^+ = \mathbf{t}\mathbf{w}_L^+ \tag{C1d}$$

Using Eqs. (C1a)~(C1d), the characteristic equation is obtained as follows

$$(\mathbf{r}_L \mathbf{t} \mathbf{r}_R \mathbf{t} - \mathbf{I})\mathbf{w}_L^+ = 0 \tag{C2}$$

where $\mathbf{I}$ denoted the 3×3 identity matrix, $\mathbf{r}_L = \mathbf{r}_R = \mathbf{r}$ and $\mathbf{t}$ are the reflection and transmission matrices expressed as

$$\mathbf{r}_{\text{hinged}} = \begin{bmatrix} -1 & -1 & -1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ -(\gamma_1 - i\alpha_1)\gamma_1 & -(\gamma_2 - i\alpha_2)\gamma_2 & -(\gamma_3 - i\alpha_3)\gamma_3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ (\gamma_1 - i\alpha_1)\gamma_1 & (\gamma_2 - i\alpha_2)\gamma_2 & (\gamma_3 - i\alpha_3)\gamma_3 \end{bmatrix} \tag{C3}$$

$$\mathbf{r}_{\text{fixed}} = \begin{bmatrix} -1 & -1 & -1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ -(i\gamma_1 + \alpha_1) & -(i\gamma_2 + \alpha_2) & -(i\gamma_3 + \alpha_3) \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ -(i\gamma_1 + \alpha_1) & -(i\gamma_2 + \alpha_2) & -(i\gamma_3 + \alpha_3) \end{bmatrix} \tag{C4}$$

$$\mathbf{r}_{\text{free}} = \begin{bmatrix} (i\gamma_1\alpha_1 - 1) & (i\gamma_2\alpha_2 - 1) & (i\gamma_3\alpha_3 - 1) \\ -(\alpha_1 + i\gamma_1)\gamma_1^2 & -(\alpha_2 + i\gamma_2)\gamma_2^2 & -(\alpha_3 + i\gamma_3)\gamma_3^2 \\ -(\gamma_1 - i\alpha_1)\gamma_1 & -(\gamma_2 - i\alpha_2)\gamma_2 & -(\gamma_3 - i\alpha_3)\gamma_3 \end{bmatrix}^{-1} \begin{bmatrix} -(i\gamma_1\alpha_1 - 1) & -(i\gamma_2\alpha_2 - 1) & -(i\gamma_3\alpha_3 - 1) \\ -(\alpha_1 + i\gamma_1)\gamma_1^2 & -(\alpha_2 + i\gamma_2)\gamma_2^2 & -(\alpha_3 + i\gamma_3)\gamma_3^2 \\ (\gamma_1 - i\alpha_1)\gamma_1 & (\gamma_2 - i\alpha_2)\gamma_2 & (\gamma_3 - i\alpha_3)\gamma_3 \end{bmatrix} \tag{C5}$$

$$\mathbf{t} = \begin{bmatrix} e^{-i\gamma_1\theta} & 0 & 0 \\ 0 & e^{-i\gamma_2\theta} & 0 \\ 0 & 0 & e^{-i\gamma_3\theta} \end{bmatrix} \tag{C6}$$

For non-trivial solution, the natural frequencies are obtained from the characteristic equation expressed as determinant.

$$C(\omega) = \det\left[\mathbf{r}_L \mathbf{t} \mathbf{r}_R \mathbf{t} - \mathbf{I}\right] = 0 \tag{C7}$$

Nansukusa Mirembe Sarah was a former graduate student at Dept. of Civil Engineering at Dong-A University. The title of her master thesis is “Frequency equation of a curved beam using the phase-closure principle”.

Hyun woo Park recent research interest is analytical, numerical and experimental investigation of high-frequency modal behaviors of a cracked beam from wave propagation perspective. He has demonstrated that the phase closure principle allows for the generic frequency equation of one-dimensional elastic waveguide with multiple incipient cracks as well.